Lecture 11

- Regular Perturbation Method
- Poincare Linstead Method.
- Uniformly Valid Approximations

Perturbation Methods

1) Regular perturbations

consider the differential equation of second order

$$F(t, y, \dot{y}, \ddot{y}, \varepsilon) = 0, \quad t \in I, \quad (1)$$

where t is the independent variable, I is an interval, and y is the dependent variable

In general, initial or boundary conditions may accompany the equation.

ε is a small parameter means

$\varepsilon << 1$.

By a perturbation series we understand a power series in the form

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \quad (2)$$

The basis of the regular perturbation method is to assume a solution of

$$F(t, y, \dot{y}, \ddot{y}, \varepsilon) = 0 \quad , \qquad t \in I \qquad (1)$$

of the form

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \qquad (2)$$

where the functions $y_0, y_1, y_2, \ldots$ are to be determined by substitution of (2) into (1). The first few terms of such series form an approximation solution, a so-called perturbation solution to the problem, usually no more than two or three forms are taken.

"The method is successful if the approximation is uniform"

This means that

"the difference of the exact and approximate solution converges to zero at some well defined rate as ε approaches zero uniformly on I."

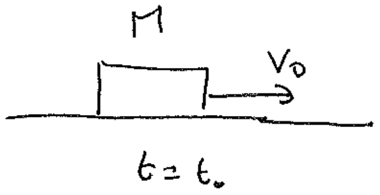
y_0 is the leading order term

y_i, $i=1,2,\ldots$, higher order terms

If the method is succesful, y_0 will be the solution of the unperturbed problem

$$F(t, y, \dot{y}, \ddot{y}, 0)=0, \quad t \in I$$

Example



$\longleftarrow f_{resistive} = av - bv^2$

$v(\tau)$ is the velocity

τ time

a, b are positive constants and $b \ll a$.

$$[a] = \frac{\text{Force}}{\text{velocity}} = \frac{\text{Mass} \times \cancel{L}/T^2}{\cancel{L}/T} = \frac{\text{mass}}{T}$$

$$[b] = \frac{\text{Force}}{\text{velocity}^2} = \frac{\text{Mass} \times \cancel{L}/T^2}{L^2/T^2} = \frac{\text{Mass}}{L}$$

$$m \frac{dv}{dt} = -av + bv^2 \tag{3}$$

define now dimensionless variables

$$y = \frac{v}{v_0}, \qquad t = \frac{\tau}{(m/a)}$$

$y(0) = 1.$

(3) becomes

$$\frac{dy}{dt} = -y + \varepsilon y^2, \quad t>0$$

Proof:

$$mV_0 \frac{dy}{d\cancel{t}} = -a V_0 y + b V_0^2 y^2$$

$$\frac{dy}{dt} \cdot \frac{mV_0}{(m/a)} = -aV_0 y + bV_0^2 y^2$$

$$\frac{dy}{dt} = -y + \frac{bV_0}{a} y^2, \quad t>0$$

let $\varepsilon = \frac{bV_0}{a} <<$

we get

$$y' = -y + \varepsilon y^2, \quad t>0$$

$y(0) = 1.$

"Bernoulli equation"

Exact solution

$$\frac{dy}{-y+\varepsilon y^2} = \frac{dy}{y(-1+\varepsilon y)} = \left[-\frac{1}{y} \quad \frac{\varepsilon}{-1+\varepsilon y}\right] dy$$

$$= 1$$

$$-\ln y + \ln(-1+\varepsilon y) = t - t_0$$

$$\frac{-1+\varepsilon y}{y} = e^{t-t_0}$$

$$-1+\varepsilon y = y e^{t-t_0}$$

$$y(\varepsilon - e^{t-t_0}) = +1$$

$$y = \frac{-1}{e^{t-t_0}-\varepsilon}$$

$$y(0) = \frac{-1}{e^{-t_0}-\varepsilon} = 1 \Rightarrow e^{-t_0} - \varepsilon = -1$$

$$e^{-t_0} = -1+\varepsilon$$

$$y(t) = \frac{-1}{e^{t}(-1+\varepsilon)-\varepsilon} = \frac{-e^{-t}}{-1+\varepsilon(1-e^{-t})}$$

$$y(t) = \frac{e^{-t}}{1+\varepsilon(e^{-t}-1)}, \quad t>0$$

exact solution is

$$y(t) = \frac{e^{-t}}{1+\varepsilon(e^{-t}-1)} , \quad t>0$$

Perturbation

$$y' = -y + \varepsilon y^2 , \quad y(0) = 1$$

$$y = y_0 + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

$$y(0) = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots = 1$$

$$y_0(0) = 1, \quad y_i(0) = 0, \quad i = 1, 2, \cdots$$

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots$$

$$= -y_0 - \varepsilon y_1 - \varepsilon^2 y_2 + \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 + \cdots)$$

ε^0: $\quad y_0' = -y_0$

ε: $\quad y_1' = -y_1 + y_0^2$

ε^2: $\quad y_2' = -y_2 + 2y_0 y_1$

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i) $y_0 = a e^{-t}$ $\qquad a = 1, \quad y_0(0) = 1$

$\underline{y_0(t) = e^{-t}}$

ii) $y_1' + y_1 = y_0^2 = e^{-2t}$

$(e^{t} y_1)' = e^{-t}$

$y_1 e^{t} = -e^{-t} + a$

$y_1 = -e^{-2t} + a e^{-t}$ $\qquad y_1(0) = 0$

$a = +1$

$\underline{y_1 = -e^{-2t} + e^{-t}}$

iii) $y_2' + y_2 = 2 y_0 y_1$

$(y_2 e^{t})' e^{-t} = 2 y_0 y_1$

$= 2(-e^{-2t} + e^{-t}) e^{-t}$

$(y_2 e^{t})' = 2(-e^{-2t} + e^{-t})$

$y_2 e^{t} = + e^{-2t} - 2e^{-t} + a$

$y_2(0) = 0$ $\qquad +1 - 2 + a = 0 \qquad a = +1$

$y_2 = + e^{-3t} - 2e^{-2t} + e^{-t}$

$= e^{-t}(+e^{-2t} - 2e^{-t} + 1) = e^{-t}(e^{-t} - 1)^2$

$$y(t) = \cancel{e^{-t} + \varepsilon(e^{-t} - 1)} + \varepsilon^2$$

$$= e^{-t} + \varepsilon\left(-e^{-2t} + e^{-t}\right)$$
$$+ \varepsilon^2\left(e^{-3t} - 2e^{-2t} + e^{-t}\right) + \varepsilon^3 \cdots$$

$$y(t) = e^{-t} \cancel{+} \varepsilon e^{-t}(e^{-t} - 1)$$
$$+ \varepsilon^2 e^{-t}(e^{-t} - 1)^2 + \cdots$$

$$y(t) = e^{-t} - \varepsilon e^{-t}(e^{-t} - 1)$$
$$+ \varepsilon^2 e^{-t}(e^{-t} - 1)^2 + \cdots$$

$$= e^{-t}\left[1 - \varepsilon(e^{-t} - 1) + \varepsilon^2(e^{-t} - 1)^2 + \cdots\right]$$

$$= e^{-t}\,\frac{1}{1 + \varepsilon(e^{-t} - 1)}$$

Consistent with the exact solution

Approximation with three term

$$y_{exact} - y_{approx.} = M_1(t)\,\varepsilon^3 + M_2(t)\,\varepsilon^4 + \cdots$$

for some bounded function

$$M_1(t) = e^{-t}(e^{-t} - 1)^3, \ \ldots \qquad t > 0$$

for a fixed value of t the

error

$$E(t, \varepsilon) = M_1(t)\varepsilon^3 + M_2(t)\varepsilon^4$$

approache to zero as $\varepsilon \to 0$

as the same rate as ε^3 goes to

zero

" It can be shown that the convergence is uniform as $\varepsilon \to 0$ in the inte $[0, \infty)$

$$|E(t, \varepsilon)| \leq M\varepsilon^3$$

$$y_{exact} - y_{app} = e^{-t}\left[1 - \varepsilon(e^{-t}-1) + \varepsilon^2(e^{-t}-1)^2 - \varepsilon^3(\ )^3 + \varepsilon^4(\ )^4 + \cdots + (-1)^n(\ )^n + \cdot\right] - e^{-t}$$

$$+ \varepsilon e^{-t}(\ ) - \varepsilon^2 e^{-t}(\ )^2\ ]$$

$$= -\varepsilon^3(e^{-t})(\ )^3 + \varepsilon^4(\ )^4 e^{-t}$$

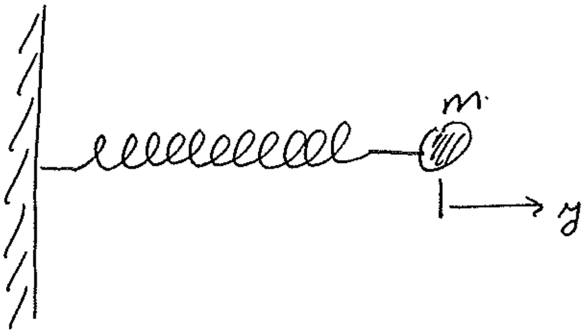
$$|y_{exact} - y_{app}| \le \varepsilon^3 e^{-t}(\ )^3\left[1 - \varepsilon(\ ) + \varepsilon^2(\ ) + \cdot\right]$$

$$\le \varepsilon^3 2^3\left[1 + 2\varepsilon + 4\varepsilon^2 + \cdots\right]$$

$$\le (2\varepsilon)^3 \frac{1}{1-2\varepsilon}$$

$$\xrightarrow{\le} 0 \text{ as } \varepsilon \to 0$$

Nonlinear Oscillator



$$m \frac{d^2y}{d\tau^2} = -ky - ay^3 \quad , \quad \tau > 0$$

$$y(0) = A \quad , \quad \frac{dy}{d\tau}(0) = 0$$

$$[k] = \frac{\text{mass}}{T^2} \quad , \quad [a] = \frac{\text{mass}}{T^2 L^2}$$

$$[m] = \text{mass}$$

$$\left[\frac{m}{k}\right] = T^2 \qquad \left[\sqrt{\frac{m}{k}}\right] = T$$

$$t = \frac{\tau}{\sqrt{m/k}} \quad , \quad u = \frac{y}{A}$$

$$\cancel{\frac{d^2y}{dt^2} = A \frac{d^2u}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 = \frac{Am}{k}}$$

$$mA \frac{d^2u}{d\tau^2} = -kA\,u - aA^3u^3$$

$$mA \frac{d^2u}{dt^2}\frac{dt^2}{d\tau^2} = mA\frac{1}{m/k}\frac{d^2u}{dt^2}$$

$$= -kAu - aA^3u^3$$

$$\frac{d^2u}{dt^2} + u + \frac{aA^2}{k}u^3 = 0$$

let $\varepsilon = \frac{aA^2}{k} \ll 1$

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$$

Duffing eqn

$$u(0) = 1, \quad \frac{du}{dt} = 0 \qquad t = 0$$

$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

$u_0(0) = 1, \quad u_i(0) = 0 \qquad t=0$

$\dfrac{du_i}{dt} = 0 \qquad t=0$

$u_0'' + u_0 = 0$

$u_1'' + u_1 + u_0^3 = 0$

$u_0 = a \cos t + b \sin t$

$u_0(0) = a = 1$

$\dfrac{du_0}{dt} = b \cos t = 0 \qquad b = 0$

$\underline{u_0(t) = \cos t}$

$\cos 3t = 4\cos^3 t - 3\cos t$

$u_1'' + u_1 = -\cos^3 t$

$= -\frac{1}{4}(\cos 3t + 3\cos t)$

$\boxed{u_p = At\cos t + Bt\sin t + C\cos 3t}$

$u_p' = A\cos t + B\sin t - At\sin t + Bt\cos t - 3C\sin 3t$

$u_p'' = -2A\sin t + 2B\cos t - At\cos t - Bt\sin t - 9C\cos 3t$

$u_p'' + u_p = \cancel{(B-2A)\sin t + (A+2B)\cos t}$

$= -2A\sin t + 2B\cos t + (C - 9C)\cos 3t$

$= -\frac{3}{4}\cos t - \frac{1}{4}\cos 3t$

$A = 0$

$B = -\frac{3}{8}$

$C = -\frac{1}{4}$

$u_p = -\frac{3}{8} t \sin t + \frac{1}{32} \cos 3t$

$u_1 = A_1 \cos t + B_2 \sin t + \frac{1}{32} \cos 3t - \frac{3}{8} t \sin t$

$u_1(0) = A_1 + \frac{1}{32} = 0$

$\frac{du_1}{dt} = B_2 = 0$

$$\boxed{u_1 = -\frac{1}{32} \cos t + \frac{1}{32} \cos 3t - \frac{3}{8} t \sin t}$$

$u_{ap} = \cos t + \varepsilon \left[\frac{1}{32} (\cos 3t - \cos t) - \frac{3}{8} t \sin t \right] + \cdots$

leading order term

$u_0 = \cos t$

The second term, or the correction term, however is not necessarily small

$$\frac{1}{32}(\cos 3t - \cos t) - \frac{3}{8} t \sin t$$

i) for a fixed value of t this term goes to ~~…~~ zero as $\varepsilon \to 0$

. but if t itself is of the order ε^{-1} or larger as $\varepsilon \to 0$, then the term $-\frac{3}{8} t \sin t$ will be large. Such a term is called a "secular term"

. Hence the amplitude of the approximate solution grows with time which is not consistent with physical expectations.

. Another fact is that the exact solution is bounded with time $t > 0$

- It is not possible to improve this by finding higher order terms, because there are secular terms of these terms as well ✓

ii) if $t \in [0, T)$ then the correction term can be made as small as possible by choosing ε sufficiently small ✓

Hence as long as, the coefficient $\frac{3\varepsilon t}{8}$ is kept small by limiting t and taking ε small the leading order term is a reasonable approximate soln.

The Poincaré–Lindstedt Method

The idea of the Poincaré–Lindstedt method is to introduce "distorted time scale in the perturbation series

In particular we set

$$u(\xi) = u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \cdots$$

where

$$\xi = \omega t$$

and

ω_0

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

$\omega_0 = 1$ is the frequency of the unperturbed solution or period is 2π

$$P = \frac{2\pi}{\omega}$$

Our equation now becomes

$\omega_0 = 1$ $$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0$$

$$\frac{d^2u}{d\xi^2}\omega^2 + u + \varepsilon u^3 = 0$$

$$u(0) = 1 \,, \quad \frac{du}{dt} = 0$$

$$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 (u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots)$$

$$+ (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$$

$$+ \varepsilon (u_0^3 + 3\varepsilon u_0^2 u_1 + \cdots) = 0$$

1) $u_0'' + u_0 = 0$

2) $u_1'' + u_1 + 2\omega_1 u_0'' + u_0^3 = 0$

3) $u_2'' + u_2 + \omega_2 u_0'' + \omega_1 u_1'' + 3u_0^2 u_1 = 0$

- $u_0 = \cos\xi$

- $u_1'' + u_1 = 2\omega_1 \cos\xi - \cos^3\xi$
$= 2\omega_1 \cos\xi - \frac{1}{4}(3\cos\xi + \cos 3\xi)$
$= (2\omega_1 - \frac{3}{4})\cos\xi - \frac{1}{4}\cos 3\xi$

Choose $2\omega_1 = 3/4$

$$u_1'' + u_1 = -\frac{1}{4}\cos 3\xi$$

$$u_1 = A\cos\xi + B\sin\xi + \frac{1}{32}\cos 3\xi$$

$$A = -\frac{1}{32} \qquad B = 0$$

$$u_1 = \frac{1}{32}\left[-\cos\xi + \cos 3\xi\right]$$

$$u(\xi) = \cos\xi + \frac{1}{32}\varepsilon\left(\cos 3\xi - \cos\xi\right) + \dots$$

$$\xi = t\left(1 + \frac{3}{8}\varepsilon + \dots\right)$$

applicable to

$$y'' + \omega_0^2 y = \varepsilon F(t, y, y'), \qquad 0 < \varepsilon \ll 1$$

$$y = y(\xi), \quad \xi = (\omega_0 + \omega_1\varepsilon + \dots)t$$

Solution of nonlinear periodic systems by Fourier Series

$V(x) = V(-x)$

$V(x) = \frac{1}{2} k_1 x^2 + \frac{1}{4} k_3 x^4 + \ldots$

$m\ddot{x} + k_1 x + k_3 x^3 + \ldots = 0$

$\ddot{x} + \alpha x + \gamma x^3 = 0$

$x(t) = b_1 \cos\omega t + b_3 \cos 3\omega t + b_5 \cos 5\omega t + \ldots$

$\dot{x} = 0$

time symmetry $\quad t \rightarrow -t$

elliptic function

1) $y = \mathrm{sn}\, u$

$$y'' + (1+k^2)\, y = 2k^2 y^3$$

2) $y = \mathrm{cn}\, u$

$$y'' + y(1-2k^2) + 2k^2 y^3 = 0$$

Exact soln

$$u_{tt} + u + \varepsilon u^3 = 0$$

$$u(t) = A\, \mathrm{cn}(at, k)$$

$$k^2 = \frac{\varepsilon A^2}{2 + 2\varepsilon A^2}$$

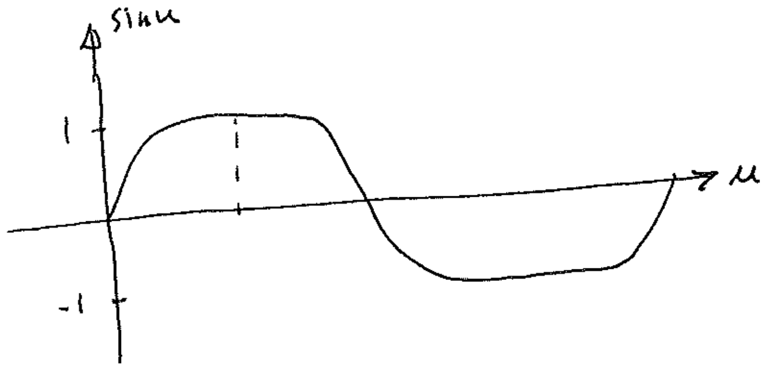
$$a^2(1 - 2k^2) = 1.$$

actual period of the motion

$$aP = 4K$$

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right]$$

$$\frac{P}{P_0} = 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2}$$



$k = 0$

$\mathrm{cn}\, u = \cos at$

Definition: Let $f(\varepsilon)$ and $g(\varepsilon)$ be defined in some neighborhood (or punctured neighborhood) of $\varepsilon = 0$. We write

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0$$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0$$

and we write

$$\{f(\varepsilon)\} = O(g(\varepsilon)), \quad \text{as} \quad \varepsilon \to 0.$$

if there exists a positive constant M such that

$$|f(\varepsilon)| \leq M \, |g(\varepsilon)|$$

for all ε in some neighborhood (punctured neighborhood) of zero.

In this definition $\varepsilon \to 0$ may be replaced by a one-sided limit or by $\varepsilon \to \varepsilon_0$ where ε_0 is any finite or infinite number with the domain of f and g defined appropriately. If $f(\varepsilon) = o(g(\varepsilon))$ holds we say f is little oh of g as $\varepsilon \to 0$, and if $f(\varepsilon) = O(g(\varepsilon))$ holds we say ~~that~~ f is big oh of g as $\varepsilon \to 0$. A common comparison function is ① $g(\varepsilon) = \varepsilon^n$ for some exponent n. Another comparison function is ② $g(\varepsilon) = \varepsilon^n \ln^m \varepsilon$ for exponents m and n.

Examples

Remark

The statement $f(\varepsilon) = o(\varepsilon)$ ~~as $\varepsilon \to 0$~~ means $f(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $f(\varepsilon) = O(1)$ means f is bounded in a neighborhood of $\varepsilon = 0$

Example Verify $\varepsilon^2 \ln \varepsilon = o(\varepsilon)$ as $\varepsilon \to 0^+$

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2 \ln \varepsilon}{\varepsilon^2} = \lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon = \lim_{\varepsilon \to 0} \frac{\ln \varepsilon}{1/\varepsilon} = \frac{1/\varepsilon}{-1/\varepsilon^2} = -\varepsilon \to 0$$

<u>Example</u> verify $\sin \varepsilon = O(\varepsilon)$ as $\varepsilon \to 0^+$

By the mean value theorem there is a number c between 0 and x such

$$\frac{\sin x - \sin 0}{x - 0} = \cos c$$

$$\sin x = x \cos c$$

Hence $|\sin x| \leq |x|$

$$|\sin \varepsilon| \leq |\varepsilon|$$

$$\sin \varepsilon = O(\varepsilon) \quad M = 1 \checkmark$$

Let $h(t, \varepsilon)$ be a function of ε and $t \in I$

we say that

$$\lim_{\varepsilon \to 0} h(t, \varepsilon) = 0 \text{ uniformly on } I$$

if the convergence to zero is at the same rate for each $t \in I$, <u>that is for any positive number η there exist</u> a positive number ε_0 independent of t such that $|h(t, \varepsilon)| < \eta$ for all $t \in I$ whenever $\varepsilon < \varepsilon_0$

Remark In an another words if $h(t,\varepsilon)$ can be made arbitrarily small over the entire I by choosing ε small enough, then the convergence is uniform.

Remark If merely $\lim_{\varepsilon \to 0} h(t_0, \varepsilon) = 0$ for each fixed $x_0 \in I$, then the convergence is called "pointwise" on I.

$\Rightarrow$ One method of proving $\lim_{\varepsilon \to 0} h(t,\varepsilon) = 0$ uniform on I is to find a function $H(\varepsilon)$ such that the inequality

$$|h(t,\varepsilon)| \leq H(\varepsilon)$$

holds for all $t \in I$, and having $H(\varepsilon) \to 0$ as $\varepsilon \to 0$. To prove that convergence is not uniform on I it is sufficient to produce a $\bar{t} \in I$ such that $|h(\bar{t},\varepsilon)| < \eta$ for some positive η, regardless how small ε is choosen.

<u>Definition</u> A function $y_a(t,\varepsilon)$ is a <u>uniformly valid asymptotic approximation</u> to a function $y(t,\varepsilon)$ on an interval I as $\varepsilon \to 0$ if the error $E(t,\varepsilon)$ defined by

$$E(t,\varepsilon) = y(t,\varepsilon) - y_a(t,\varepsilon)$$

converges to zero as $\varepsilon \to 0$ uniformly on I

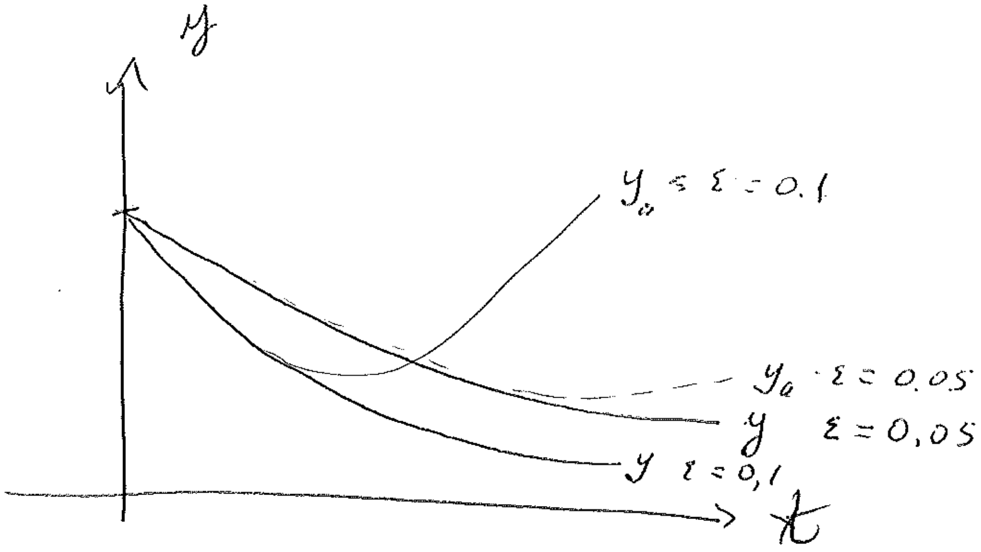
$$E(t,\varepsilon) = o(\varepsilon^n) \quad \text{or} \quad O(\varepsilon^n)$$

<u>Example</u>: $y(t,\varepsilon) = e^{-t\varepsilon} \qquad t>0, \ \varepsilon \ll 1$

$$y_a(t,\varepsilon) = 1 - t\varepsilon + \tfrac{1}{2}\varepsilon^2 t^2$$

$$E(t,\varepsilon) = e^{-t\varepsilon} - 1 + t\varepsilon - \tfrac{1}{2}\varepsilon^2 t^2 \qquad t>0$$

$\lim_{\varepsilon \to 0} E(t,\varepsilon) = 0$ for all $t \in [0,T]$ uniform but not on $t \in [0,\infty)$



Comparison of $y(t,\varepsilon)$ (solid)

and $y_0(t,\varepsilon)$ (dashed)

for $\varepsilon = 0.1$, $\varepsilon = 0,05$

for larger ε deviation

is larger.

$$y - 1 = u$$

$$\dot{u} + \varepsilon u + (\varepsilon - 1) e^{-1-u} = 0$$

$$\frac{du}{\varepsilon u + \frac{(\varepsilon-1)}{e} e^{-u}} = -dt$$

$$\frac{e^{u}\, du}{\varepsilon u e^{u} + \frac{\varepsilon - 1}{e}} = -dt$$

Remark

1. If we don't know the exact solution (usually we don't) and then a direct Error function estimate can not be done.

2. Therefore we require some notion of how well an approximate solution satisfies the differential equation and the auxiliary condition

<u>Definition</u> We say that an approximate solution $y_a(t,\varepsilon)$ satisfies the differential equation in ($F(t,y,\dot{y},\ddot{y})=0$) uniformly for $t \in I$ as $\varepsilon \to 0$ if

$$r(t;\varepsilon) = F(t, y_a(t,\varepsilon), \dot{y}_a(t,\varepsilon), \ddot{y}_a(t,\varepsilon)) \to 0$$

uniformly on I as $\varepsilon \to 0$

We can regard $r(t,\varepsilon)$ as the residual error.

Example

$$\ddot{y} + \dot{y}^2 + \varepsilon y = 0 \qquad t > 0, \qquad 0 < \varepsilon << 1$$

$$y(0) = 0, \qquad \dot{y}(0) = 1$$

i) $y(t,\varepsilon) = y_0(t,\varepsilon) + \varepsilon y_1(t,\varepsilon) + \ldots$

$$\ddot{y}_0 + \varepsilon \ddot{y}_1 + (\dot{y}_0 + \varepsilon \dot{y}_1 + \ldots)^2 + \varepsilon y_0 + \varepsilon^2 y_1 + \ldots = 0$$

$$y_0(0) = 0, \quad \dot{y}_0(0) = 1$$

$$y_i(0) = 0 \quad \dot{y}_i(0) = 0 \qquad i = 1, 2, \ldots$$

i) $\ddot{y}_0 + \dot{y}_0^2 = 0 \qquad y_0(0) = 0, \quad \dot{y}_0(0) = 1$

$$y_0 = \ln(t+1)$$

ii) $\ddot{y}_1 + 2\dot{y}_1 \dot{y}_0 + y_0 = 0$

$$\ddot{y}_1 + \frac{2}{1+t} \dot{y}_1 + y_0 = 0$$

$$\left[(1+t)^2 \dot{y}_1\right]^{\cdot} + (1+t)^2 \ln(1+t) = 0$$

$y = y_0$

$r(t,\varepsilon) = \varepsilon \ln(t+1)$

if $t \in [0,T]$ then

$$\lim_{\varepsilon \to 0} r(t,\varepsilon) = 0$$

$$(\dot{y} e^{y})^{\cdot} + \varepsilon y e^{y} = 0$$

$$\dot{y} e^{y} + \varepsilon (y e^{y} - e^{y}) + A = 0$$

$$\dot{y} + \varepsilon (y - 1) + A e^{-y} = 0$$

$$\dot{y}(0) = 1 \,,\quad y(0) = 0$$

$$1 + \varepsilon(-1) + A = 0 \qquad A = \varepsilon - 1$$

$$\dot{y} + \varepsilon y - \varepsilon + (\varepsilon - 1) e^{-y} = 0$$

$$(e^{\varepsilon t} y)^{\cdot} e^{-\varepsilon t} - \varepsilon + (\varepsilon - 1) e^{-y} = 0$$

$$(e^{\varepsilon t} y)^{\cdot} + (\varepsilon - 1) e^{\varepsilon t} e^{-y} - \varepsilon e^{\varepsilon t} = 0$$

$$\int (1+t)^2 \ln(1+t)\, dt$$

$$= \int u^2 \ln u \, du = \frac{1}{3} u^3 \ln u - \frac{1}{3} \int u^3 \cdot \frac{1}{u}\, du$$

$$= \frac{1}{3} u^3 \ln u - \frac{1}{9} u^3 + c$$

$$(1+t)^2 \dot{y}_1 + \frac{1}{3} (1+t)^2 \ln(1+t) - \frac{1}{9} (1+t)^3 + C = 0$$

$$\dot{y}_1 + \frac{1}{3} \ln(1+t) - \frac{1}{9} (1+t) + \frac{C}{(1+t)^2} = 0$$

$$y_1 + \frac{1}{3} \left[(1+t) \ln(1+t) - (1+t) \right] - \frac{1}{2} (1+t)^2$$

$$- \frac{C}{1+t} + D = 0$$

$$y_1(0) + \frac{1}{3} (-1) - \frac{1}{2} - C + D = 0$$

$$D = C + \frac{5}{6}$$

$$y_1'(0) + \frac{1}{9} [-1] + C = 0 \qquad C = +\frac{1}{9}$$

$$D = \frac{1}{9} + \frac{5}{6} = \frac{2+15}{18} = \frac{17}{18}$$

$$y_1 + \frac{1}{3} \left[(1+t) \ln(1+t) - (1+t) \right] - \frac{1}{2} (1+t)^2 - \frac{17}{18(1+t)} + \frac{1}{9} = 0$$

SET 9

MATH 543: PERTURBATION

Reference: David Logan.

About the uniform convergence of a perturbation series we have mainly the following three definitions

Definition 1: *Let $f(t,\varepsilon)$ and $g(t,\varepsilon)$ be defined for all $t \in I$ in some neighborhood (or punctured neighborhood) of $\varepsilon = 0$. We write*

$$f(t,\varepsilon) = o(g(t,\varepsilon)) \text{ as } \varepsilon \to 0$$

if

$$\lim_{\varepsilon \to 0} \left| \frac{f(t,\varepsilon)}{g(t,\varepsilon)} \right| = 0$$

pointwise on I. If the limit above is uniform on I, we write $f(t,\varepsilon) = o(g(t,\varepsilon))$ uniformly on I. If there exists a positive function $M(t)$ on I such that

$$|f(t,\varepsilon)| \leq M(t)|g(t,\varepsilon)|$$

for all $t \in I$ and ε in some neighborhood of zero, then we write

$$f(t,\varepsilon) = O(g(t,\varepsilon)) \text{ as } \varepsilon \to 0, \text{ uniformly on } I$$

Definition 2. *A function $y_a(t,\varepsilon)$ is a uniformly valid approximation to a function $y(t,\varepsilon)$ on an interval I as $\varepsilon \to 0$ if the error function $E(t,\varepsilon)$ defined by*

$$E(t,\varepsilon) = y(t,\varepsilon) - y_a(t,\varepsilon)$$

converges to zero as $\varepsilon \to 0$ uniformly for all $t \in I$

Definition 3. *Let $F(t, y(t), y'(t), \cdots, \varepsilon)$ be a differntial equation. We say that an approximate solution $y_a(t, \varepsilon)$ satisfies the differential equation uniformly for $t \in I$ as $\varepsilon \to 0$ if the residual error function $r(t, \varepsilon)$ goes to zero*

$$r(t, \varepsilon) = F(t, y_a(t, \varepsilon), y_a'(t, \varepsilon), \cdots, \varepsilon) \to 0$$

uniformly on I.

Problems:

(1) Let $y' + y - \varepsilon y^2 = 0$ for $t > 0$ with $y(0) = 1$. Find approximate solutions (up to second order perturbation for instance) and discuss whether the approximate solution is a uniformly valid solution.

(2) Let $y'' + y + \varepsilon y^3 = 0$ for $t > 0$ with $y(0) = 1, \;\; y'(0) = 0$. Solve this equation up to second order in ε.

(3) Let $y'' + \varepsilon(y')^2 + y = 0$ for $t > 0$ with $y(0) = 1, \;\; y'(0) = 0$. Find a first order approximation of this problem and discuss the validity of the approximation.

(4) Let $y'' - y = \varepsilon t y$ for $t > 0$ with $y(0) = 1, \;\; y'(0) = -1$. Find a first order approximation of the boundary value problem and discuss the validity of the approximation.

(5) Use the Poincare'-Lindstedt method to obtain a two term (first order perturbation) perturbation approximation to the following problems:

(i) $y'' + y = \varepsilon y(y')^2, \;\; y(0) = 1, \;\; y'(0) = 0.$

(ii) $y'' + 9y = 3\varepsilon y^3, \;\; y(0) = 0, \;\; y'(0) = 1.$

(iii) $y'' + y = \varepsilon y(1 - y'^{\,2}, \;\; y(0) = 1, \;\; y'(0) = 0$

(6) Find a two term perturbation solution of

$$y' + y = \frac{1}{1 + \varepsilon y},$$

for $t > 0$ with $y(0) = 0$ and $\varepsilon << 1$. Compare with the exact solution and discuss the validity of the approximation.